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The quantum mechanical geometric phase of a particle in a resonant vibrating cavity

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Abstract

We study the general-setting quantum geometric phase acquired by a particle in a vibrating cavity. Solving the two-level theory with the rotating-wave approximation and the $SU(2)$ method, we obtain analytic formulae that give excellent descriptions of the geometric phase, energy and wavefunction of the resonating system. In particular, we observe a sudden π -jump in the geometric phase when the system is in resonance. Similar behaviour is found in the geometric phase of a spin- $\frac{1}{2}$ particle in a rotating magnetic field. Our result suggests that the π -jump is a general feature for two-level systems, for which we developed a geometrical model to help visualize its evolution.

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1. Introduction

The dynamics of confined cavity fields interacting with the cavity wall is of great interest for the understanding of a variety of problems such as hadron bag models [1], sonoluminescence [2], cavity QED [3] and black hole radiation [4]. Most previous studies on dynamical cavities concentrated on scalar or photon fields [5], which despite the apparent simplicity, exhibit rich and complex dynamics. In contrast, the system of a Schrödinger particle in an oscillating spherical cavity has not gained as much attention. In this paper, we study the generalized quantum geometric phase of a particle in resonance with the vibrating cavity. We will show that the geometric phase acquires sudden π -jumps when the particle makes transitions.

The geometric phase of a quantum system has drawn much attention since it was first proposed by Berry in 1984 [6, 7]. It reveals the geometric structure of the Hilbert space of the system [7, 8], and its existence in physical systems has been verified in many experiments [9], including electron diffraction from a screw dislocation [10] and neutron interferometry [11]. The phase effects in molecular physics can lead to energy splittings and shift quantum numbers. The geometric phase has also been shown to be intimately connected to the physics of fractional statistics, the quantized Hall effect, anomalies in gauge theory [7] and neutrino

oscillation [12]. As far as we know, our study represents the first calculation of the geometric phase of a resonating system, which evolves non-adiabatically and non-cyclically.

2. Formalism

We consider an infinite cylindrical or spherical potential well with oscillating boundary [13]:

$$V(r) = \begin{cases} 0 & \text{if } r < R(t) \\ \infty & \text{if } r \geq R(t) \end{cases} \quad (2.1)$$

where $R(t) = R_0/\alpha(t)$, with $\alpha(t) \equiv [1 + \epsilon \sin(\omega t)]^{-1}$. This particular oscillation of the cavity has been chosen for simplicity. We will show below, especially in light of our rotating-wave approximations (RWA) results, that the geometric phase depends not on the particular form of the oscillation, but on whether it contains resonating terms. The coordinates can be transformed to a fixed domain via $\vec{y} \equiv \alpha(t)\vec{r}$, and, to preserve unitarity, the wavefunction is renormalized through $\phi(\vec{y}, t) \equiv \alpha^{-\xi} \psi(\vec{r}, t)$, where $\xi = 1$ (3/2), for a cylindrical (spherical) well.

Since the full Hamiltonian $H(t)$ commutes with L^2 and \vec{L} , the wavefunction can be factorized: $\phi(\vec{y}, t) = Y(y, t)\Theta$, where Θ depends only on the angular variables. Inside the cavity, the radial wavefunction satisfies

$$\frac{\partial Y}{\partial t} = \left[\frac{i\hbar\alpha^2}{2\mu} \left(\frac{\partial^2}{\partial y^2} + \frac{n_d}{y} \frac{\partial}{\partial y} - \frac{m_d}{y^2} \right) + \frac{\dot{R}}{R} \left(y \frac{\partial}{\partial y} + \xi \right) \right] Y \quad (2.2)$$

where μ is the particle mass, and $n_d = 1, 2$ and $m_d = m^2, l(l+1)$ for cylindrical and spherical wells, respectively. In this paper, we only consider the $m_d = 0$ sector.

The wavefunction described by equation (2.2) evolves in time and acquires a time-dependent phase, which in general consists of a dynamical phase and a geometric one [6]. When the dynamics is adiabatic and/or the evolution is cyclic, the geometric phase, or Berry's phase, has been studied for many systems [14, 15]. Since we are interested in the geometric phase of a non-adiabatic, non-cyclic system, we have to resort to a generalized method. Following [16], we first remove the dynamical phase from the wavefunction of the system: $\phi(t) \rightarrow \tilde{\phi}(t) \equiv e^{-i\theta(t)}\phi(t)$, the dynamical phase is

$$\theta(t) \equiv -\frac{1}{\hbar} \int_0^t E(t') dt' \quad (2.3)$$

where $E(t') = \text{Re}\langle\phi(t')|H(t')|\phi(t')\rangle$. Note that the wavefunction $\phi(t)$ fulfils the Schrödinger equation with the full Hamiltonian $H(t)$.

The general-setting geometric phase, or the Pancharatnam phase β , is defined as the relative phase between the state at time t_1 and that at time t , both with the dynamical phase removed. It can be obtained from the inner product of these two states

$$\beta(t_1, t) \equiv -i \log \left[\frac{\langle\tilde{\phi}(t_1)|\tilde{\phi}(t)\rangle}{|\langle\tilde{\phi}(t_1)|\tilde{\phi}(t)\rangle|} \right]. \quad (2.4)$$

Our main goal in this paper is to study this geometric phase for a simple but nontrivial dynamical resonating system. If the particle is off resonance, the geometric phase acquired is trivial.

3. Results

We have solved equation (2.2) numerically by the Crank–Nicholson method [17] and checked that the solution is stable and converges very well. For example, the energy changes by only about 0.05% when the time step δt of 0.001 is reduced by half. In figure 1, we show

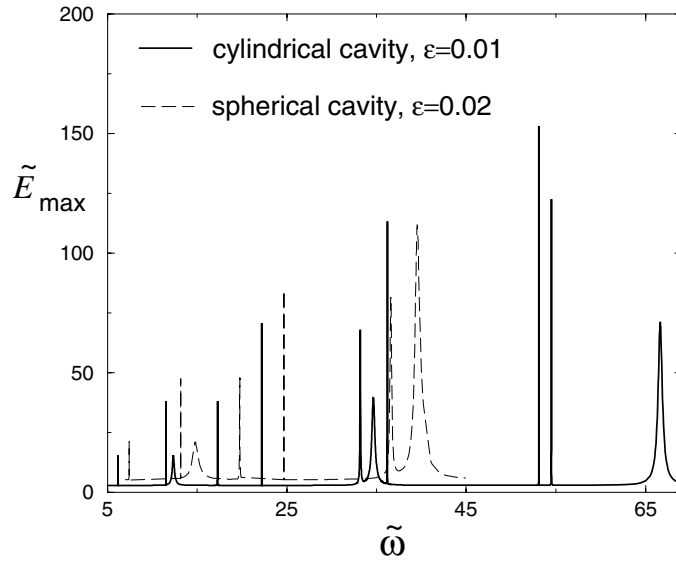


Figure 1. The maximum energy of the particle as a function of the driving frequency, obtained by solving the Schrödinger equation numerically, for a cylindrical cavity with $\epsilon = 0.01$ (solid line) and a spherical cavity with $\epsilon = 0.02$ (dashed line).

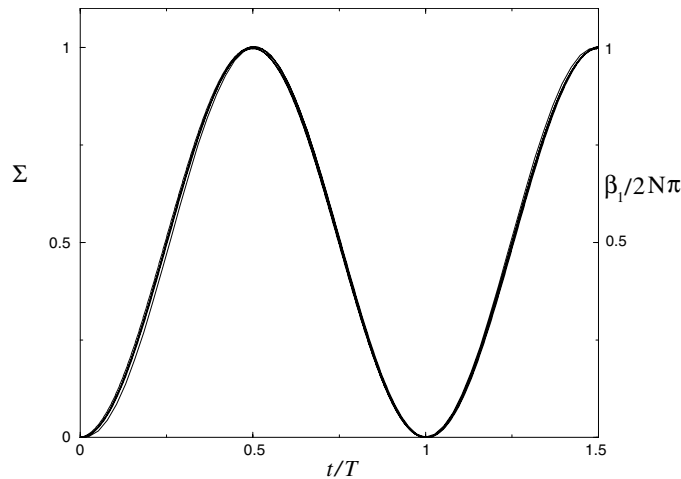


Figure 2. The scaled energy, $\Sigma \equiv (E\alpha^{-2} - E_k)/(E_n - E_k)$ where E_k and E_n are the unperturbed eigenenergies, and geometric phase acquired at each cycle, $\beta_1/2N\pi$, versus t/T , T being the oscillation period of the geometric phase, obtained by the numerical method, for the case of cylindrical cavity at $\tilde{\omega} = 66.632, 17.278$ and 22.21227 , using $\epsilon = 0.01$. The numerical results are shown together with both the RWA and $SU(2)$ approximations, and they are virtually indistinguishable.

the maximum energy of the particle as a function of the driving frequency ω , both having been scaled to be dimensionless: $\tilde{E}_{\max} \equiv \mu R_0^2 E_{\max}/\hbar^2$, and $\tilde{\omega} \equiv \mu R_0^2 \omega/\hbar$. The particle is initially in the unperturbed ground state, and it is in resonance at specific values of $\tilde{\omega}$. In each period of cavity vibration $\tau \equiv 2\pi/\tilde{\omega}$, the geometric phase acquires some changes, $\beta_1(t_1) \equiv \beta(t_1, t_1 + \tau)$, and it exhibits oscillations at these resonances. We show three examples in figure 2 for $\tilde{\omega} = 66.632, 17.278$ and 22.21227 , where the geometric phase acquires

periodic changes of 2π , 4π and 6π , respectively. All the resonances shown in figure 1 are associated with oscillations of β_1 with amplitudes of $2N\pi$, $N = 1, 2, 3$.

To understand the resonances and the associated geometric phases, we limit ourselves to the parameter regime where the Hamiltonian can be truncated to a two-level system. Specifically, if $\epsilon \ll 1$, and the initial state is the k th unperturbed eigenstate $|\phi_k(y)\rangle$, with eigenenergy E_k , then when the driving frequency corresponds to the energy difference between the initial state and another unperturbed eigenstate $|\phi_n(y)\rangle$, with eigenenergy E_n , i.e. $\omega = \omega_{nk} \equiv (E_n - E_k)/\hbar$, the particle is expected to behave as in a two-level system. The full Hamiltonian $H(t)$ under the two-level approximation is

$$\sum_{a,b=n,k} |\phi_a\rangle\langle\phi_a|H(t)|\phi_b\rangle\langle\phi_b| = \begin{pmatrix} \alpha^2(t)E_k & -i\hbar\eta_{nk}\dot{R}/R \\ i\hbar\eta_{nk}\dot{R}/R & \alpha^2(t)E_n \end{pmatrix} \quad (3.1)$$

where

$$\eta_{nk} \equiv \langle\phi_n| \left(y \frac{\partial}{\partial y} + \xi \right) |\phi_k\rangle \quad (3.2)$$

is a constant that depends only on the states involved.

Then the problem simplifies considerably, and we have obtained its solution with two analytic approaches: the $SU(2)$ method and the rotating-wave approximation (RWA).

Following Cheng and Fung [18] we first expand the time-dependent Hamiltonian $H(t)$ of a two-level system in the identity operator I , the raising and lowering operators σ_{\pm} and Pauli spin matrix σ_3

$$H(t) = i[f_0(t)I + f_1(t)\sigma_+ + f_2(t)\sigma_3 + f_3(t)\sigma_-] \quad (3.3)$$

where f_j , $j = 0, 1, 2, 3$, are in general complex functions of time. In our case, comparing equations (3.1) and (3.3) we have

$$f_0(t) = -i(E_n + E_k)\alpha^2(t)/2 \quad (3.4)$$

$$f_1(t) = -f_3(t) = -\hbar\eta_{nk}\dot{R}/R \quad (3.5)$$

$$f_2(t) = -i\hbar\omega_{nk}\alpha^2(t)/2. \quad (3.6)$$

The evolution operator can then be written as

$$U(t, 0) = b(t) e^{\int_0^t f_0(t')dt'} \begin{pmatrix} b^{-2}(t) + g_1(t)g_3(t) & g_1(t) \\ g_3(t) & 1 \end{pmatrix} \quad (3.7)$$

where $b(t) \equiv \exp(-g_2(t))$, and g_i satisfies

$$\dot{g}_1 = f_1 + 2f_2g_1 - f_3g_1^2 \quad (3.8)$$

$$\dot{g}_2 = f_2 - f_3g_1 \quad (3.9)$$

$$\dot{g}_3 = f_3b^{-2} \quad (3.10)$$

with the initial conditions $g_j(0) = 0$ for $j = 1, 2$ or 3 .

Suppose that the two levels in which the system oscillates are $|\phi_k(y)\rangle$ and $|\phi_n(y)\rangle$, so that the wavefunction is $|\phi(y, t)\rangle = a_k(t)|\phi_k(y)\rangle + a_n(t)|\phi_n(y)\rangle$. Furthermore, if the initial conditions are $a_k(0) = 1$ and $a_n(0) = 0$, then

$$\begin{pmatrix} a_k(t) \\ a_n(t) \end{pmatrix} = b(t) e^{\int_0^t f_0(t')dt'} \begin{pmatrix} b^{-2} + g_1g_3 \\ g_3 \end{pmatrix}. \quad (3.11)$$

Within the two-level approximation regime, the $SU(2)$ method is exact and reduces the problem to solving the ordinary differential equations for g_j .

The spirit of the RWA is to retain only those terms in the Hamiltonian that correspond to the resonance frequency. We first separate out the fast phase factors from the wavefunction:

$$|\phi(y, t)\rangle = c_k(t) e^{-i\rho_k(t)} |\phi_k(y)\rangle + c_n(t) e^{-i\rho_n(t)} |\phi_n(y)\rangle \quad (3.12)$$

with $\rho_i(t) \equiv E_i \int_0^t \alpha^2(t') dt' / \hbar$. Substituting equation (3.12) in the Schrödinger equation with the truncated Hamiltonian equation (3.3), we have

$$\begin{pmatrix} \dot{c}_k \\ \dot{c}_n \end{pmatrix} = \eta_{nk} \begin{pmatrix} 0 & -W^*(t) \\ W(t) & 0 \end{pmatrix} \begin{pmatrix} c_k \\ c_n \end{pmatrix} \quad (3.13)$$

where

$$W(t) \equiv \frac{\dot{R}(t)}{R(t)} \exp \left[i\omega_{nk} \int_0^t \alpha^2(t') dt' \right]. \quad (3.14)$$

As ϵ is small, we can expand $W(t)$ as a series of ϵ . Retaining all terms up to third order, we have:

$$\begin{aligned} W(t) = \epsilon e^{i\omega_{nk}t} & \left\{ \omega \cos \omega t + \epsilon \left[i\omega_{nk} (\cos 2\omega t + 1) - \frac{\omega}{2} \sin 2\omega t \right] \right. \\ & + \frac{\epsilon^2}{4\omega} \left[(\omega^2 - 6\omega_{nk}^2) \cos \omega t - (\omega^2 + 2\omega_{nk}^2) \cos 3\omega t \right. \\ & \left. \left. - i\frac{7}{2}\omega_{nk}\omega (\sin \omega t + \sin 3\omega t) \right] \right\}. \end{aligned} \quad (3.15)$$

Note that $W(t)$ consists of oscillatory terms with various frequencies depending on ω . In the spirit of RWA, we keep only terms with the *lowest* frequency for each ω , the rationale being that c_k and c_n vary slowly in time and contributions to W from high frequency terms cancel on average over such a long time scale. It is clear then from equation (3.15) that W is largest if $\omega = \omega_{nk}/N$ with N an integer, because of the emergence of zero frequency terms. For these driving frequencies, W is large and effective in inducing transitions, and we have resonances. At or close to a resonance, equation (3.15) simplifies tremendously, and closed-form solutions for equation (3.13) can be obtained easily.

For example, when $\delta\omega \equiv \omega - \omega_{nk} \approx 0$, $W(t) \approx \epsilon\omega \exp(-i\delta\omega t/2)/2$, and we have Rabi oscillations [19]:

$$c_k(t) = e^{i\delta\omega t/2} [\cos \chi t - i\delta\omega \sin \chi t / (2\chi)] \quad (3.16)$$

$$c_n(t) = \Gamma e^{-i\delta\omega t/2} \sin \chi t / \chi \quad (3.17)$$

where we took $c_k(0) = 1$, $c_n(0) = 0$, $\chi \equiv \sqrt{\Gamma^2 + \delta\omega^2/4}$ and $\Gamma = \Gamma_1 \equiv \eta_{nk}\epsilon\omega/2$. The maximum value of $|c_n|^2$, $|c_n|_{\max}^2(\omega) = 1/[1 + (\delta\omega/2\Gamma)^2]$, is a Lorentzian with a full width at half maximum (FWHM) of 4Γ .

The solutions for higher N resonances differ only in the widths $\Gamma = \Gamma_N \equiv \epsilon^N \eta_{nk} \gamma_N / 2N$, where

$$\gamma_N = \begin{cases} (\omega_{nk} + \delta\omega) & N = 1 \\ (3\omega_{nk} - \delta\omega)/4 & N = 2 \\ \frac{17\omega_{nk}^2 - 17\omega_{nk}\delta\omega + 2\delta\omega^2}{24(\omega_{nk} + \delta\omega)} & N = 3 \end{cases} \quad (3.18)$$

Table 1. Scaled widths ($\tilde{\Gamma} \equiv \mu R_0^2 \Gamma / \hbar \epsilon^N \eta_{nk}$) of the lowest $k = 1$ resonances for a cylindrical cavity with $\epsilon = 0.01$. Full numerical results are compared to the RWA values. The widths multiplied by the Rabi oscillation period, both extracted from the numerical data, are also shown in the last column. Corresponding numbers for a spherical cavity with $\epsilon = 0.02$ are shown in parentheses.

Resonances		$\tilde{\Gamma}$		$T\Gamma/\pi$
N	n	Numerical	RWA	Numerical
1	2	6.17 (7.45)	6.17 (7.40)	1.00 (1.01)
1	3	16.6	17.3	0.98
1	4	33.2	33.3	1.00
2	2	4.15 (4.66)	4.63 (5.55)	0.99 (0.96)
2	3	50.0 (76.5)	13.0 (14.8)	1.00 (1.02)
2	4	891 (532)	25.0 (27.8)	1.09 (1.17)
2	5	136	40.7	0.93
3	4	205 (341)	23.6 (26.2)	1.02 (1.07)
3	5	15 700	38.5	1.02
3	6	78 300	56.8	1.07

with $\delta\omega \equiv N\omega - \omega_{nk}$. The FWHM of the resonances $4\Gamma/N \propto \epsilon^N$ are narrower for larger N , and the oscillation periods $T \propto \epsilon^{-N}$ are longer.

The energy of the system is given in RWA by

$$E(t) \approx \alpha^2(t) [E_k \cos^2 \chi t + (A + E_n) \sin^2 \chi t] \quad (3.19)$$

where $A \equiv [E_k \delta\omega^2 + 4E_n(\Gamma^2 - \chi^2)]/4\chi^2$. The energy of the particle therefore exhibits fast but small oscillations due to the α^2 factor, modulated by a large but slow oscillation of period π/χ . It follows that

$$\frac{(1 - \epsilon)^2 E_{\max} - E_k}{E_n - E_k} \approx \frac{1}{1 + (\delta\omega/2\Gamma)^2} = |c_n|_{\max}^2(\omega) \quad (3.20)$$

which is again a universal Lorentzian for all resonances.

The RWA results are virtually identical with those of the $SU(2)$ method. Figure 1 can be understood completely in terms of overlapping series of the ω_{nk}/N resonances, and the positions of the peaks are in almost exact agreement with the RWA or $SU(2)$ predictions. The resonance line shapes are well fitted by the Lorentzian equation (3.20), though the widths are underpredicted for the $N = 2, 3$ resonances, as shown in table 1. For example, the ($N = 2, n = 4$) resonance has a much larger width than predicted by RWA and $SU(2)$, which is mainly due to the involvement of other states during transition, such as a ($1 \rightarrow 3 \rightarrow 2$) process, which is second order in perturbation theory and so affects the $N > 1$ transitions more severely. Occasionally, resonances at similar frequencies may overlap and lead to broadened widths.

At a resonance, the Rabi oscillations of the particle as predicted in RWA (equation (3.19)) can be explicitly seen in figure 2, where the energy of the particle versus time, calculated numerically, is shown for $n = 4, 3, 4$ and $N = 1, 2, 3$. The RWA also predicts that at exact resonances, $\chi = \Gamma$, and therefore the oscillation periods should be inversely proportional to the widths of the resonances, $T = \pi/\chi = \pi/\Gamma$. As shown in the last column of table 1, this is clearly borne out in the numerical data, where $T\Gamma/\pi$ is listed and is found to be close to 1 for all resonances, deviating by less than 10% for even those with strong mixing.

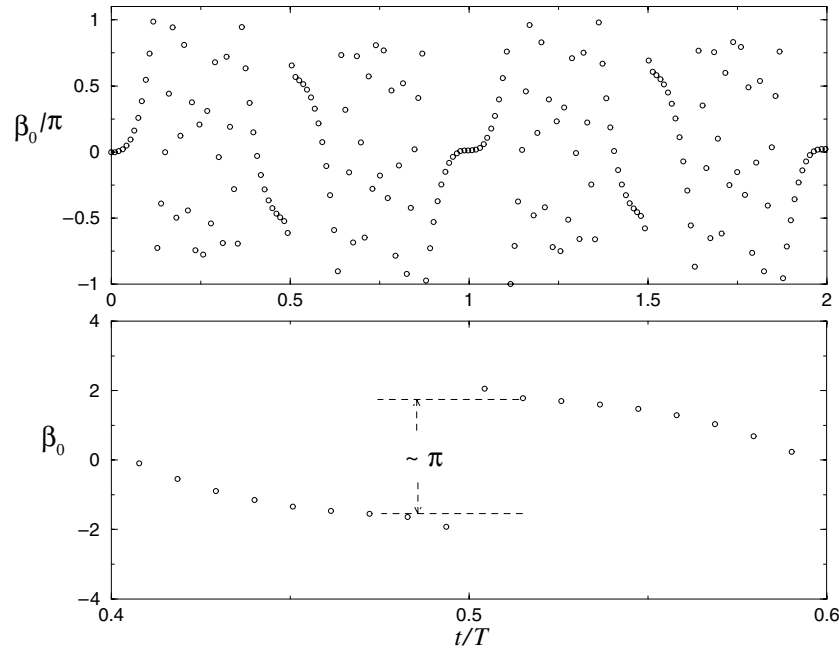


Figure 3. β_0 versus t/T at $\tilde{\omega} = 12.344$ for a cylindrical cavity with $\epsilon = 0.01$. The region around $t/T = 0.5$ is enlarged and shown in the lower panel to show the approximate π -jump. Numerical results are shown here.

Using the RWA and two-level approximation, we can calculate the geometric phase easily. Removing the dynamical phase with the help of equation (3.19), we have

$$|\tilde{\phi}(t)\rangle \approx e^{i\omega'_{nk} \int_0^t \sin^2(\chi t') dt'} \begin{pmatrix} c_k(t) \\ c_n(t) e^{-i\omega_{nk} t} \end{pmatrix} \quad (3.21)$$

where $\omega'_{nk} \equiv A/\hbar + \omega_{nk}$. For $\omega = \omega_{nk}/N$ resonances, if we choose $t - t_1$ to be an integral multiple, q , of the cavity oscillation period τ , we get

$$\beta(t_1, t_1 + q\tau) = \frac{\omega'_{nk}}{2\chi} [\tilde{\chi} - \sin \tilde{\chi} \cos(2\chi t_1 + \tilde{\chi})] \quad (3.22)$$

where $\tilde{\chi} \equiv \chi q\tau$. In particular, if $t_1 = 0$,

$$\beta_0(q\tau) \equiv \beta(0, q\tau) = \begin{cases} \frac{\Omega(q\tau)}{2} & \text{for } (2m - \frac{1}{2}) < t/T < (2m + \frac{1}{2}) \\ \frac{\Omega(q\tau)}{2} \pm \pi & \text{for } (2m + \frac{1}{2}) < t/T \leq (2m + \frac{3}{2}) \end{cases} \quad (3.23)$$

where $\Omega(t) = \omega_{nk} t - \frac{\omega_{nk}}{2\chi} \sin 2\chi t$, with $m = 0, 1, 2, \dots$. There are sudden approximate π -jumps in β_0 at $t/T = (2m + 1/2)$, which are indeed seen in the numerical results shown in figure 3, such as, for example, at $\tilde{\omega} = 12.344$. Since $\chi\tau \ll 1$, the phase change in each cycle ($q = 1$) is

$$\beta_1(t_1) \equiv \beta(t_1, t_1 + \tau) \approx \omega'_{nk} \tau \sin^2 \chi(t_1 + \tau/2). \quad (3.24)$$

At an exact N th resonance, $\delta\omega = 0$ and $A = 0$, and so

$$\beta_1(t_1) \approx 2N\pi \sin^2[\chi(t_1 + N\pi/\omega_{nk})]. \quad (3.25)$$

Therefore, the geometric phase oscillates with an amplitude of $2N\pi$ and period of T . Both of these RWA predictions are in excellent agreement with the numerical data, as shown in

figure 2. Note that the π -jumps and the functional forms of the geometric phases are independent of ϵ and \hbar , as long as they are nonzero. Also note that while the geometric phase equation (2.4) is not defined when the overlap $\langle \tilde{\phi}(t_1) | \tilde{\phi}(t) \rangle$ is zero, which happens, for example, at $t/T = 2m + \frac{1}{2}$ in equation (3.23) and figure 3, precisely where the π -jumps occur, it is well defined and calculable everywhere else, and therefore there is no problem identifying and calculating the π -jumps.

In summary, even though all the results we presented in figures 1–3 are from numerical computation, the $SU(2)$ and RWA approximations are very good near most resonances (see also table 1) and help us to understand these results. In particular, the analytic formulae derived using RWA give excellent descriptions of the geometric phase, energy and wavefunction of the resonating system.

4. A neutron in a magnetic field

To gain more insight into the geometric phase for a two-level system, we have studied a simple model of a magnetic field rotating around a spin- $\frac{1}{2}$ particle. Suppose a neutron with gyromagnetic ratio γ is placed at the origin, in the presence of a magnetic field

$$\mathbf{B}(t) = B_0[\sin \alpha \cos(\omega t) \hat{i} + \sin \alpha \sin(\omega t) \hat{j} + \cos \alpha \hat{k}] \quad (4.1)$$

which has a constant magnitude B_0 but its direction sweeps out a cone with an opening angle α , $0 < \alpha < \pi$, at a constant angular speed ω . The Hamiltonian of the system is given by

$$H(t) = -\gamma \mathbf{B} \cdot \mathbf{S} = -\frac{\hbar \omega_1}{2} \begin{pmatrix} \cos \alpha & e^{-i\omega t} \sin \alpha \\ e^{i\omega t} \sin \alpha & -\cos \alpha \end{pmatrix} \quad (4.2)$$

where $\omega_1 \equiv \gamma B_0$ and \mathbf{S} is the spin matrix.

The system can be solved analytically [20]. The instantaneous eigenspinors of $H(t)$ with eigenenergies $E_+ = -E_- = -\hbar \omega_1/2$, respectively, are

$$|\psi_+(t)\rangle = \begin{pmatrix} \cos(\alpha/2) \\ e^{i\omega t} \sin(\alpha/2) \end{pmatrix} \quad (4.3)$$

and

$$|\psi_-(t)\rangle = \begin{pmatrix} \sin(\alpha/2) \\ -e^{i\omega t} \cos(\alpha/2) \end{pmatrix}. \quad (4.4)$$

Suppose the neutron spin is initially parallel to $\mathbf{B}(0)$, and we consider the case when the particle makes a transition to spin down along the instantaneous direction of \mathbf{B} . This happens with unit probability if $\omega = -\omega_1/\cos \alpha$, provided that $\cos \alpha \neq 0$. The state vector at any time is then [20]

$$|\psi(t)\rangle = e^{-i\omega t/2} \left\{ \cos\left(\frac{\lambda t}{2}\right) |\psi_+(t)\rangle + i \sin\left(\frac{\lambda t}{2}\right) |\psi_-(t)\rangle \right\} \quad (4.5)$$

where $\lambda = \omega \sin \alpha$.

The Pancharatnam phases comparing the initial state with the state at time t for different values of α are shown in figures 4–6. The sudden π -jump and a two-period oscillation can be seen in the case when $\alpha = 1/p$, p being an even integer, such as the case in figure 4. When p is an odd integer, the Pancharatnam phase performs a single-period oscillation with no π -jump, as shown in figure 5. It has neither single-period nor two-period oscillation when other values of α are used, figure 6. If $t = q\tau$, with q an integer and $\tau = 2\pi/\omega$,

$$\beta_0(q\tau) = \begin{cases} -\frac{\Omega(q\tau)}{2} & \text{for } (2m - \frac{1}{2}) < q\tau/T \leq (2m + \frac{1}{2}) \\ -\frac{\Omega(q\tau)}{2} \pm \pi & \text{for } (2m + \frac{1}{2}) < q\tau/T \leq (2m + \frac{3}{2}) \end{cases} \quad (4.6)$$

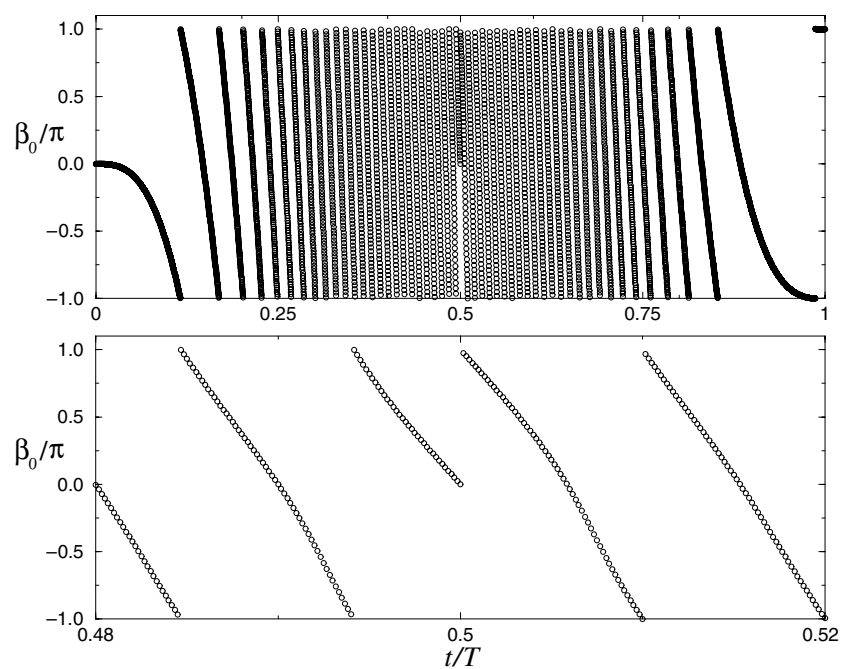


Figure 4. Same as figure 3, but for a spin- $\frac{1}{2}$ particle in a rotating magnetic field, $\alpha = 1/100$, and for $t \neq n\tau$.

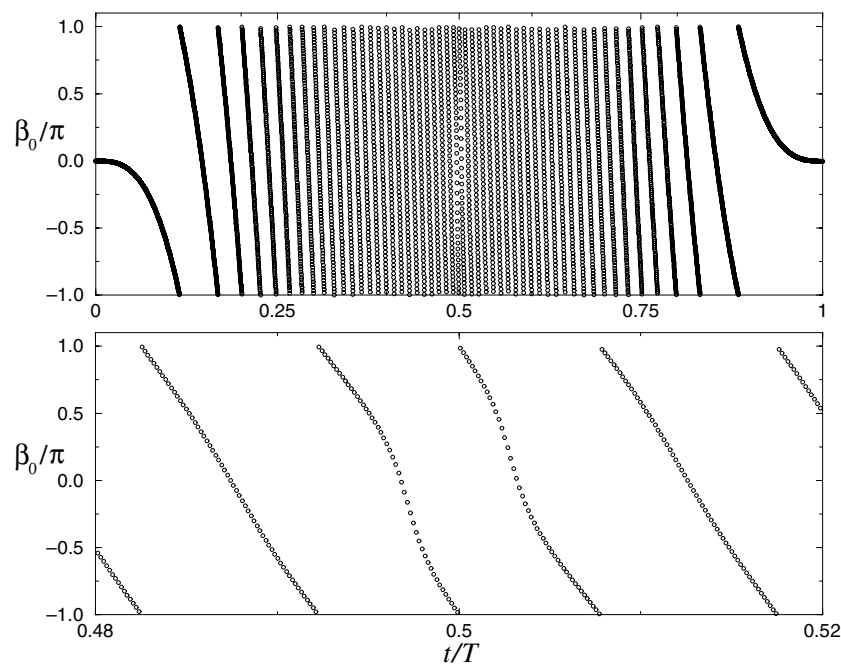


Figure 5. Same as figure 4, but for $\alpha = 1/101$.

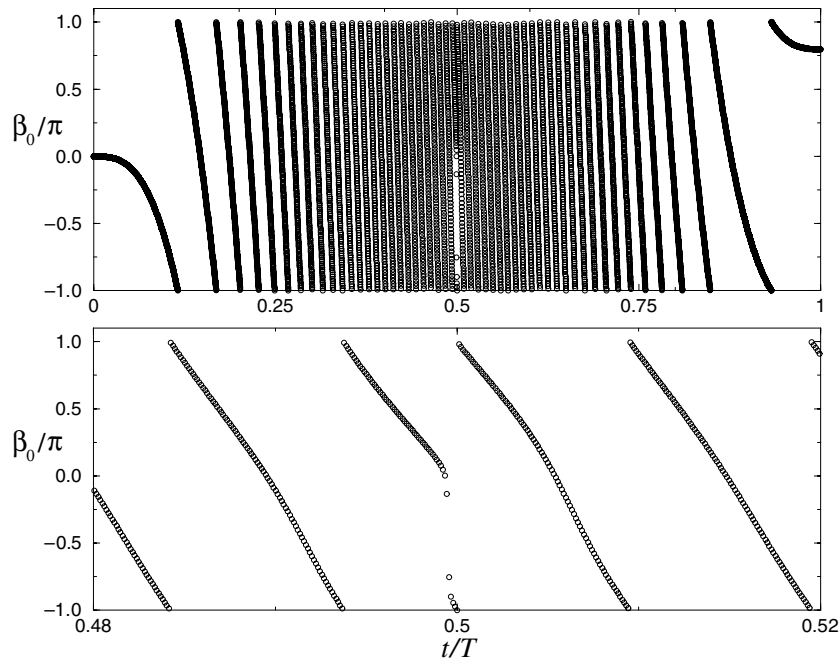


Figure 6. Same as figure 4, but for $\alpha = 5/501$.

which has the same form as equation (3.23), where m is an integer, $T = 2\pi/\lambda$, and

$$\Omega(t) \equiv \left(\frac{\omega_1}{\lambda} \sin \lambda t + \omega t \right). \tag{4.7}$$

For the case $t_1 = (q - 1)\tau$ and $t = q\tau$,

$$\beta_1(t_1) = \begin{cases} -\frac{1}{2}[\Omega(t_1 + \tau) - \Omega(t_1)] & \text{for } 0 < \sin \alpha < \frac{1}{2} \\ -\frac{1}{2}[\Omega(t_1 + \tau) - \Omega(t_1)] \pm \pi & \text{for } \frac{1}{2} < \sin \alpha < 1. \end{cases} \tag{4.8}$$

In the limit $\alpha \ll 1$,

$$\beta_1(t_1) \approx -2\pi \sin^2 \left(\frac{\lambda}{2}(t_1 + \tau/2) \right), \tag{4.9}$$

which has the same form as equation (3.25).

In the cyclic limit, i.e. $t = q\tau = T \equiv 2\pi/\lambda$,

$$\Omega(T) = \omega T = \omega q\tau = 2q\pi \tag{4.10}$$

and thus the geometric phases are $\beta_0(T) = -(q - 1)\pi$ and $\beta_1(T) = -\pi$.

Since the geometric phases in the two models—a neutron in a rotating magnetic field and a particle in a vibrating cavity—are remarkably similar, we conjecture that the main features of the generalized geometric phase we calculated, especially the π -jumps, are universal for a particle in transition from one state to another in two-level systems. Similar features about the geometric phase have also been obtained in [21].

We can use a geometrical model based on Bloch’s sphere picture [22] to help visualize the evolution of the geometric phase of a two-level system. It is clear from equation (4.5) that the state vector traces out a path on the unit sphere defined by the angular coordinates $\theta \equiv \lambda t$

and $\phi \equiv \omega t$. The excitation condition of our system therefore gives us the trajectory of the state on this unit sphere,

$$\theta = \phi \sin \alpha \quad (4.11)$$

which is a spiral curve. In our case, we have used a time-dependent basis $|\psi_+(t)\rangle, |\psi_-(t)\rangle$, which mark the north and south poles, respectively. The solid angle subtended at the origin by the spiral curve and the geodesic, $\phi = 0$, up to any $t = q\tau$ is

$$\Omega_o = \int_0^{2q\pi} \int_0^{\phi' \sin \alpha} \sin \theta' d\theta' d\phi' \quad (4.12)$$

$$= 2q\pi - \frac{\sin(2q\pi \sin \alpha)}{\sin \alpha} \quad (4.13)$$

which coincides with equation (4.7), in the limit $\alpha \ll 1$. Therefore, we see from equations (4.6) and (4.8) that the geometric phases β_0 and β_1 are simply the solid angles subtended by the spiral curve, $\theta = \alpha\phi$, from $\phi = 0$ and $\phi = 2(q-1)\pi$ up to $\phi = 2q\pi$, respectively, divided by -2 . This picture is a generalization of Berry's [6] for adiabatic and cyclic evolution corresponding to trajectories with constant $\theta = \theta_0$, and the solid angle is simply $\Omega_o = 2\pi(1 - \cos \theta_0)$.

5. Summary

In summary, we have reported the first calculation of the quantum geometric phase of a physical system in resonance, that of a particle in a vibrating cylindrical or spherical cavity, and we have shown that it acquires sudden π -jumps when the particle makes transitions from one state to another. We have derived analytic expressions in the RWA and $SU(2)$ methods, which give an excellent description of the energy, wavefunctions and the geometric phases at these resonances. We found remarkably similar properties of the geometric phases for the simple system of a neutron in a rotating magnetic field, which led us to conjecture that the main features of the generalized geometric phases and especially the π -jump we found are universal for two-level systems. We have also developed a geometrical model to help visualize these phases.

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